

On the dynamics of a spherically shaped top on a plane with friction

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Abstract

The simple realistic model of the tippe top is considered. An averaged system of equations of motion is obtained in special evolutionary variables. Through the qualitative analysis of this system the general features of the motion of the top are obtained. Finally, some numerical results are presented.

1.Introduction

In this paper we investigate the motion of a top of spherical shape on a plane with sliding friction. Such a problem has been considered in a large number of studies concerning the dynamics of the tippe top - a children's toy with the paradoxical behavior (the complete bibliography can be found in [1] - [7]). Following O'Braien and Synge [3], we postulate a viscous friction law with the force of friction proportional to the velocity of sliding and opposed to that velocity. Although this assumption has been doubted in [6], it allows us to explain and qualitatively predict the top's possible behavior.

The analysis will be restricted to motions which are close to regular precessions. In order to write equations of motion in the form convenient for application of asymptotic methods we shall use the special evolutionary variables suggested in [8, 9].

2.Equations of motion

Let us consider a spherically shaped top of radius r placed on horizontal plane Π (Fig.1). The mass distribution inside the top is axially symmetrical. The mass center G of the top lies at a distance a from the geometrical center of the top's surface. At the contact point P the normal reaction force \mathbf{N} and the force of friction \mathbf{F} are applied to the top. As mentioned in Sect.1, we shall assume that the force \mathbf{F} is proportional to the velocity of the point P .

To describe the equations of motion we introduce several Cartesian coordinate systems. The system $OXYZ$ is a spatially fixed coordinate system with the axis OZ directed upward; the plane OXY coincides with Π . The coordinate system $\widetilde{G}\widetilde{X}\widetilde{Y}\widetilde{Z}$

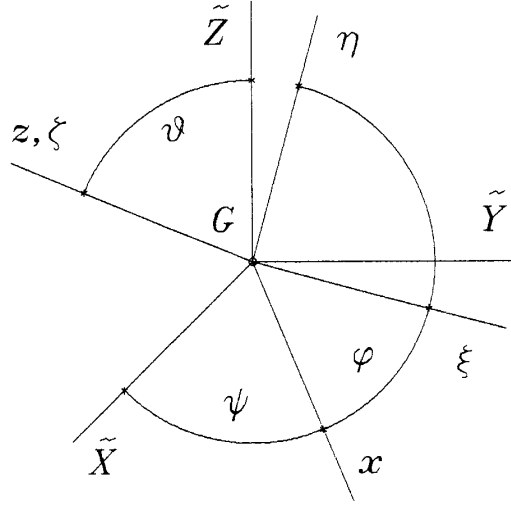


Figure 2: The coordinate systems used

are given by formulae

$$\begin{aligned} V_{PX} &= V_{GX} - \cos \psi [(r \cos \vartheta - a) \Omega_y + r \Omega_z \sin \vartheta] - \sin \psi (r - a \cos \vartheta) \Omega_x, \\ V_{PY} &= V_{GY} - \sin \psi [(r \cos \vartheta - a) \Omega_y + r \Omega_z \sin \vartheta] + \cos \psi (r - a \cos \vartheta) \Omega_x. \end{aligned} \quad (2.3)$$

in which $\Omega_x, \Omega_y, \Omega_z$ are the projections of the top's angular velocity vector ω onto the axes of the semifixed coordinate system $Oxyz$.

The magnitude of the normal reaction force \mathbf{N} can be expressed as

$$N = |\mathbf{N}| = m[g + a(\Omega_x^2 \cos \vartheta + \dot{\Omega}_x \sin \vartheta)].$$

Here g is constant of gravity.

The equations for the motion of the top about the center of mass have the following form

$$\begin{aligned} (A + ma^2 \sin^2 \vartheta) \dot{\Omega}_x &= -(C\Omega_z - A\Omega_y \operatorname{ctg} \vartheta) \Omega_y + \kappa \sin \vartheta [1 + \cos \vartheta (a\Omega_x^2/g)] + M_x, \\ A\dot{\Omega}_y &= (C\Omega_z - A\Omega_y \operatorname{ctg} \vartheta) \Omega_x + M_y, \quad C\dot{\Omega}_z = M_z, \\ \dot{\psi} &= \Omega_y / \sin \vartheta, \quad \dot{\vartheta} = \Omega_x, \quad \dot{\varphi} = \Omega_z - \Omega_y \operatorname{ctg} \vartheta. \end{aligned} \quad (2.1b)$$

Here C and A are axial and central transverse moments of inertia, κ is equal to the product mga , M_x, M_y, M_z are the projections of the torque caused by the friction force onto the axes of the semifixed system. Taking into account the relations (2.2), (2.3) we find

$$\begin{aligned} M_x &= -\varepsilon(r - a \cos \vartheta) NV_{P*}, \\ M_y &= \varepsilon(r \cos \vartheta - a) NV_{Px}, \quad M_z = -\varepsilon r \sin \vartheta NV_{Px}, \\ V_{Px} &= V_{GX} \cos \psi + V_{GY} \sin \psi - (r - a \cos \vartheta) \Omega_y + r \Omega_z \sin \vartheta. \end{aligned}$$

$$V_{P*} = -V_{GX} \sin \psi + V_{GY} \cos \psi + (r - a \cos \vartheta) \Omega_x ,$$

Dynamical equations (2.1) have Jellet's integral

$$L = A\Omega_y \sin \vartheta + (r \cos \vartheta - a)C\Omega_z = C_\circ .$$

We shall suppose below that

$$\begin{aligned} \varepsilon \kappa^{3/2} A^{-1/2} &\ll 1 , \quad |\omega| \sim (\kappa/A)^{1/2} , \\ (V_{GX}^2 + V_{GY}^2)^{1/2} &\sim r(\kappa/A)^{1/2} . \end{aligned} \quad (2.4)$$

When relations (2.4) take place, the influence of the friction on the motion of the top can be treated as a certain weak perturbation.

Under the proper choice of time, mass and length scales the sliding friction coefficient ε will be a small parameter in equations (2.1). It allows us to investigate the dynamics of the top on the plane by means of asymptotic methods.

3.Special evolutionary variables

First let us consider some properties of the unperturbed motion. For $\varepsilon = 0$ system (2.1) describes the motion of the top along an absolutely smooth surface and has the first integrals

$$V_{GX} = C_1 , \quad V_{GY} = C_2 , \quad (3.1a)$$

and

$$u \equiv C\Omega_z = C_3 , \quad w \equiv A\Omega_y \sin \vartheta + C\Omega_z \cos \vartheta = C_4 , \quad (3.1b)$$

$$E = \frac{1}{2} \left[m \left(V_{GX}^2 + V_{GY}^2 \right) + \left(A + ma^2 \sin^2 \vartheta \right) \Omega_x^2 + A\Omega_y^2 + C\Omega_z^2 \right] - \kappa \cos \vartheta = C_5$$

Here u and w are the projections of the angular momentum onto the axis of symmetry of the top and onto the vertical axis respectively, E denotes the total energy of the top.

In the unperturbed case subsystem (2.1b) governing the rotational motion of the top reduces to the form

$$\begin{aligned} (A + ma^2 \sin^2 \vartheta) \dot{\Omega}_x &= -(C\Omega_z - A\Omega_y \operatorname{ctg} \vartheta) \Omega_y + \kappa \sin \vartheta [1 + (a\Omega_x^2/g)] , \\ A\dot{\Omega}_y &= (C\Omega_z - A\Omega_y \operatorname{ctg} \vartheta) \Omega_x , \quad C\dot{\Omega}_z = 0 , \\ \dot{\varphi} &= \Omega_y / \sin \vartheta , \quad \dot{\vartheta} = \Omega_x , \quad \dot{\psi} = \Omega_z - \Omega_y \operatorname{ctg} \vartheta . \end{aligned} \quad (3.2)$$

Equations (3.2) are integrable by quadratures [10, 11]. In general, in the unperturbed motion, the quantity Ω_z is constant, Ω_x, Ω_y and ϑ are periodic functions of t with period T_ϑ , ψ and φ can be expressed as follows

$$\psi = \omega_\psi t + \tilde{\psi}(t) , \quad \varphi = \omega_\varphi t + \tilde{\varphi}(t) .$$

Here $\tilde{\psi}(t)$ and $\tilde{\varphi}(t)$ are T_ϑ -periodic functions of t . The frequencies $\omega_\vartheta = 2\pi/T_\vartheta$, ω_ψ and ω_φ depend in a complicated manner on the values of the first integrals (3.1b) and in general are incommensurable.

System (3.2) has a two-parameter family of stationary solutions

$$\begin{aligned} \Omega_x &\equiv 0 , \quad \Omega_y \equiv \Omega_{y_0} , \quad \Omega_z \equiv \Omega_{z_0} , \quad \vartheta \equiv \Theta , \\ \psi &= Wt + \psi_0 , \quad \varphi = \omega_{\varphi_0} t + \varphi_0 . \end{aligned} \quad (3.3)$$

The constants ψ_0 and φ_0 in (3.3) are arbitrary, while $\Omega_{y_0}, \Omega_{z_0}, \omega_{\varphi_0}, W$ and Θ are connected by the relations

$$\begin{aligned} \Omega_{y_0} &= W \sin \Theta , \quad C\Omega_z = -\frac{\kappa}{W} + AW \cos \Theta , \\ \omega_{\varphi_0} &= \frac{1}{C} \left[(A - C)W \cos \Theta - \frac{\kappa}{W} \right] . \end{aligned}$$

Solutions (3.3) correspond to those motions which can be represented by a certain superposition of a uniform rotation about the axis of symmetry and a uniform rotation about the vertical. Such motions are called “regular precessions”. It is convenient to choose the velocity of the precession W and the angle of nutation Θ as the parameters of the family (3.3).

A closed subsystem of equations for Ω_x, Ω_y and ϑ can be derived from (3.3), containing Ω_z as a parameter. Setting

$$\Omega_z = \frac{1}{C} \left(-\frac{\kappa}{W} + AW \cos \Theta \right) . \quad (3.4)$$

we consider an integral manifold $S_{W,\Theta}$ in the phase space $(\Omega_x, \Omega_y, \vartheta)$ with a fixed value for the integral w , pertaining to the regular precession with the parameters W and Θ [12]. Its parametric representation has the form

$$\begin{aligned} S_{W,\Theta} &= \{ (\Omega_x, \Omega_y, \vartheta) : \Omega_x = \Omega_x(W, \Theta, c, \nu) , \quad \Omega_y = \Omega_y(W, \Theta, c, \nu) , \\ &\quad \vartheta = \vartheta(W, \Theta, c, \nu) ; \quad 0 \leq \nu \leq 2\pi , \quad 0 \leq c \leq c_0(W, \Theta) \} \end{aligned}$$

where c and ν denote the amplitude and the phase of the nutational oscillations. At individual solution lying on the manifold $S_{W,\Theta}$ $\nu = \omega_\vartheta t + \nu_0$. It is not difficult to prove, through Lyapunov’s holomorphic integral theorem [13], that the functions $\Omega_x(W, \Theta, c, \nu)$, $\Omega_y(W, \Theta, c, \nu)$, $\vartheta(W, \Theta, c, \nu)$ can be written in the form of the series

$$\begin{aligned} \Omega_x &= \sum_{k=1}^{\infty} c^k \Omega_{xk}(W, \Theta, \nu) , \quad \Omega_y = \Omega_{y_0} + \sum_{k=1}^{\infty} c^k \Omega_{yk}(W, \Theta, \nu) , \\ \vartheta &= \Theta + \sum_{k=1}^{\infty} c^k \vartheta_k(W, \Theta, \nu) . \end{aligned} \quad (3.5)$$

which converge for sufficiently small values of $|c|$ (to apply Lyapunov's theorem it is necessary to reduce the order of the system for $\Omega_x, \Omega_y, \vartheta$ using the integral w). We have the following expressions for the first coefficients

$$\Omega_{x1} = \omega_o \sin \nu, \quad \Omega_{y1} = -(\kappa \cos \nu)/AW, \quad \vartheta_1 = \cos \nu.$$

Here $\omega_o = \sqrt{(A^2 W^4 + 2\kappa A W^2 \cos \Theta + \kappa^2)/AW^2(A + ma^2 \sin^2 \Theta)}$ is the frequency of the small nutational oscillations.

The formulae (3.4),(3.5) define the local change of variables

$$(\Omega_x, \Omega_y, \Omega_z, \vartheta) \longrightarrow (W, \Theta, c, \nu)$$

The new variables have a simple mechanical meaning: W and Θ specify the reference regular precession, while c and ν characterize the amplitude and phase of the nutational oscillations in motion which is close to the reference precession. It is implied that this motion and the reference precession belong to the same joint level of the integrals u and w .

This change of variables reduces system (2.1) to a form which is convenient for the application of the averaging method [14].

Variables W, Θ and c are independent integrals of unperturbed system. The following relations hold

$$u = -\frac{\kappa}{W} + AW \cos \Theta, \quad w = -\frac{\kappa \cos \Theta}{W} + AW. \quad (3.6)$$

4. Equations of motion of the top in the special evolutionary variables

At first, we obtain equations for the variables W, Θ by means of two sequential substitutions:

$$(\Omega_y, \Omega_z) \xrightarrow{1} (u, w) \xrightarrow{2} (W, \Theta).$$

For $\varepsilon \neq 0$ the change in the projections of the angular momentum onto the symmetry axis and onto the vertical is described by the equations

$$\dot{u} = M_z, \quad \dot{w} = M_z \cos \vartheta + M_y \sin \vartheta. \quad (4.1)$$

Expressing u and w in (4.1) in terms of W and Θ in accordance with (3.6), we find

$$\frac{\partial u}{\partial W} \dot{W} + \frac{\partial u}{\partial \Theta} \dot{\Theta} = -\varepsilon r \sin \vartheta N V_{Px},$$

$$\frac{\partial w}{\partial W} \dot{W} + \frac{\partial w}{\partial \Theta} \dot{\Theta} = -\varepsilon a \sin \vartheta N V_{Px} . \quad (4.2)$$

Equations (4.2) define a system of linear equations for \dot{W} and $\dot{\Theta}$ with the determinant

$$D = \frac{\partial(u, w)}{\partial(W, \Theta)} = \omega_{\circ}^2 \sin \Theta A (A + m a^2 \sin^2 \Theta) / W$$

System (4.2) can be solved if $W \neq 0$ and $\sin \Theta \neq 0$:

$$\begin{aligned} \dot{W} &= -\varepsilon \sin \vartheta N V_{Px} \frac{\partial L}{\partial \Theta} \frac{\partial(W, \Theta)}{\partial(u, w)} , \\ \dot{\Theta} &= \varepsilon \sin \vartheta N V_{Px} \frac{\partial L}{\partial W} \frac{\partial(W, \Theta)}{\partial(u, w)} . \end{aligned} \quad (4.3)$$

The substitution $(\Omega_x, \vartheta) \longrightarrow (c, \nu)$ is analogous to the Van der Pol substitution [14]. Slightly modifying the Van der Pol approach, we find

$$\begin{aligned} \dot{c} &= \frac{\varepsilon N}{\Delta_{\circ}} \left[\frac{-(r - a \cos \vartheta) V_{P*}}{A + m a^2 \sin^2 \vartheta} \frac{\partial Q}{\partial \nu} + \sin \vartheta V_{Px} \left(\Delta_W^c \frac{\partial L}{\partial \Theta} - \Delta_{\Theta}^s \frac{\partial L}{\partial W} \right) \frac{\partial(u, w)}{\partial(W, \Theta)} \right] , \\ \dot{\nu} &= \omega_{\vartheta} - \frac{\varepsilon N}{\Delta_{\circ}} \left[\frac{-(r - a \cos \vartheta) V_{P*}}{A + m a^2 \sin^2 \vartheta} \frac{\partial Q}{\partial c} + \sin \vartheta V_{Px} \left(\Delta_W^{\nu} \frac{\partial L}{\partial \Theta} - \Delta_{\Theta}^{\nu} \frac{\partial L}{\partial W} \right) \frac{\partial(u, w)}{\partial(W, \Theta)} \right] , \end{aligned}$$

Here $Q(W, \Theta, c, \nu) = \vartheta - \Theta$ and functions $\Delta_{\circ}, \Delta_W^c, \Delta_{\Theta}^c, \Delta_W^{\nu}, \Delta_{\Theta}^{\nu}$ are defined by formulae

$$\begin{aligned} \Delta_{\circ} &= \frac{\partial \omega_{\vartheta}}{\partial c} \left(\frac{\partial Q}{\partial \nu} \right)^2 - \omega_{\vartheta} \left(\frac{\partial Q}{\partial c} \frac{\partial^2 Q}{\partial \nu^2} - \frac{\partial Q}{\partial \nu} \frac{\partial^2 Q}{\partial c \partial \nu} \right) , \\ \Delta_W^c &= \frac{\partial \omega_{\vartheta}}{\partial W} \left(\frac{\partial Q}{\partial \nu} \right)^2 - \omega_{\vartheta} \left(\frac{\partial Q}{\partial W} \frac{\partial^2 Q}{\partial \nu^2} - \frac{\partial Q}{\partial \nu} \frac{\partial^2 Q}{\partial W \partial \nu} \right) , \\ \Delta_{\Theta}^c &= \frac{\partial \omega_{\vartheta}}{\partial \Theta} \left(\frac{\partial Q}{\partial \nu} \right)^2 - \omega_{\vartheta} \left[\left(1 + \frac{\partial Q}{\partial \Theta} \right) \frac{\partial^2 Q}{\partial \nu^2} - \frac{\partial Q}{\partial \nu} \frac{\partial^2 Q}{\partial \Theta \partial \nu} \right] , \\ \Delta_W^{\nu} &= \frac{\partial Q}{\partial \nu} \left(\frac{\partial \omega_{\vartheta}}{\partial W} \frac{\partial Q}{\partial c} - \frac{\partial \omega_{\vartheta}}{\partial \nu} \frac{\partial Q}{\partial W} \right) - \omega_{\vartheta} \left(\frac{\partial Q}{\partial W} \frac{\partial^2 Q}{\partial c \partial \nu} - \frac{\partial Q}{\partial c} \frac{\partial^2 Q}{\partial W \partial \nu} \right) , \\ \Delta_{\Theta}^{\nu} &= \frac{\partial Q}{\partial \nu} \left(\frac{\partial \omega_{\vartheta}}{\partial \Theta} \frac{\partial Q}{\partial c} - \frac{\partial \omega_{\vartheta}}{\partial \nu} \frac{\partial Q}{\partial \Theta} \right) - \\ &\quad \omega_{\vartheta} \left(\frac{\partial Q}{\partial \Theta} \frac{\partial^2 Q}{\partial c \partial \nu} - \frac{\partial Q}{\partial c} \frac{\partial^2 Q}{\partial \Theta \partial \nu} \right) - \frac{\partial}{\partial c} \left(\omega_{\vartheta} \frac{\partial Q}{\partial \nu} \right) . \end{aligned}$$

We will not write down awkward equations for $\dot{\psi}, \dot{\varphi}, \dot{V}_{GX}, \dot{V}_{GY}$ in terms of special evolutionary variables. We only note that in general we have

$$\dot{\nu} = O(1), \quad \dot{\psi} = O(1), \quad \dot{\varphi} = O(1)$$

while

$$\dot{V}_{GX} = O(\varepsilon), \quad \dot{V}_{GY} = O(\varepsilon), \quad \dot{W} = O(\varepsilon), \quad \dot{\Theta} = O(\varepsilon), \quad \dot{c} = O(\varepsilon c).$$

Clearly, $V_{GX}, V_{GY}, W, \Theta, c$ are slow variables, ν, ψ, φ are fast variables.

5. Averaged equations

We will analyze the behavior of the slow variables $V_{GX}, V_{GY}, W, \Theta, c$ by the averaging method in the version developed by Volosov [15]. Because the right hand sides of the equations of motion do not depend on the proper rotation angle φ , the averaging along the unperturbed motion reduces to the independent averaging with respect to ν and ψ (in a nonresonant case). The averaging procedure is complicated by the unevenness of the variation of the fast variable ψ due to the presence of the periodic component. However, taking into account arguments similar to those contained in [16], this difficulty is easy to overcome.

In the first approximation of the averaging method we find (we are retaining the previous notation for the averaged variables)

$$\begin{aligned} \dot{V}_{GX} &= -\varepsilon g V_{GX} (1 + O(c^2)), \quad \dot{V}_{GY} = -\varepsilon g V_{GY} (1 + O(c^2)), \\ \dot{W} &= -\varepsilon m g \sin \Theta U \frac{\partial L}{\partial \Theta} \frac{\partial(W, \Theta)}{\partial(u, w)} + O(\varepsilon c^2), \\ \dot{\Theta} &= \varepsilon m g \sin \Theta U \frac{\partial L}{\partial W} \frac{\partial(W, \Theta)}{\partial(u, w)} + O(\varepsilon c^2), \\ \dot{c} &= \varepsilon m g c \left[\Xi_1 + (\Xi_2 + U \Xi_3) \frac{\partial(W, \Theta)}{\partial(u, w)} \right] + O(\varepsilon c^2) \end{aligned} \tag{5.1}$$

Here $U = \sin \Theta [r \Omega_{z_0}(W, \Theta) - (r \cos \Theta - a)W]$ is the averaged projection of the absolute velocity of the point P on the Gx axis in the regime of a regular precession of the top at a velocity W with a nutation angle Θ , functions $\Xi_1(\Theta), \Xi_2(W, \Theta), \Xi_3(W, \Theta)$ are given by

$$\begin{aligned} \Xi_1 &= -\frac{(r - a \cos \Theta)^2}{2(A + m a^2 \sin^2 \Theta)}, \\ \Xi_2 &= \sin \Theta \left[\frac{L}{2A} - (r - a \cos \Theta)W \right], \end{aligned}$$

$$\Xi_3 = \frac{\sin \Theta}{4\omega_o^2} \frac{\partial(\omega_o^2, L)}{\partial(W, \Theta)} - [\cos \Theta - (ma^2\omega_o^2/2\kappa) \sin^2 \Theta] \frac{\partial L}{\partial W} .$$

As well as the original system (2.1), the averaged equations have Jellet's integral

$$L(W, \Theta) = rw - au = C_o .$$

6. Qualitative analysis of the motion of the top on the basis of the averaged equations

The analysis of the averaged equations (5.1) reveals a weak interaction between rotational and horizontal motions of the top. Indeed, in the equations for $\dot{V}_{GX}, \dot{V}_{GY}$ the influence of the rotational motion is expressed in negligibly small terms of order εc^2 . Omitting these terms, we find

$$V_{GX}(t) = V_{GX}(0)e^{-\varepsilon g t}, \quad V_{GY}(t) = V_{GY}(0)e^{-\varepsilon g t} . \quad (6.1)$$

Thus the horizontal component of the center mass velocity asymptotically goes to zero. The characteristic time of the top's braking is $T_{br} = 1/\varepsilon g$.

Since the equations for $\dot{W}, \dot{\Theta}, \dot{c}$ compose a closed subsystem, the evolution of the rotational motion does not depend on the horizontal motion at all. In the phase space (W, Θ, c) the condition $c = 0$ defines an integral manifold. Solutions lying on the manifold tend formally to regular precessions when $\varepsilon \rightarrow 0$. These solutions have been investigated in [1, 2, 7]. Since there are no terms linear in c in the equations for \dot{W} and $\dot{\Theta}$, small nutational oscillations influence the behavior of the variables W and Θ in a weak manner.

The phase portrait of the above-mentioned subsystem on the manifold $c = 0$ gives us a clear picture of the general properties of the rotational motion of the top. As an example, Fig.3 shows phase portraits constructed for a top with parameters $r/a = 5$, $ma^2/A = 0.09$. The trajectories are the levels of Jellet's integral $L(W, \Theta) = C_o$. Since the phase portraits are symmetric with respect to the axis $W = 0$, the figures can be restricted to the regions of positive values of the precession velocity.

Taking into account the sign of the expression in the square brackets in the last equation (5.1), we can define the attracting ($[\cdot \cdot \cdot] < 0$) and repelling ($[\cdot \cdot \cdot] > 0$) regions on the integral manifold $c = 0$. On phase portraits the repelling regions are shaded. If the trajectory drifts above the attracting (repelling) region near the manifold $c = 0$ the amplitude of small nutational oscillations decreases (increases) (Fig.4).

Now let us consider stationary motions of the top on the horizontal plane with friction. By setting in (5.1)

$$\dot{V}_{GX} = \dot{V}_{GY} = \dot{W} = \dot{\Theta} = \dot{c} = 0$$

we can deduce the following relations:

$$V_{GX} = V_{GY} = c = 0, \quad \cos \Theta = \frac{aCW^2 - r\kappa}{rW^2(C - A)} . \quad (6.2)$$

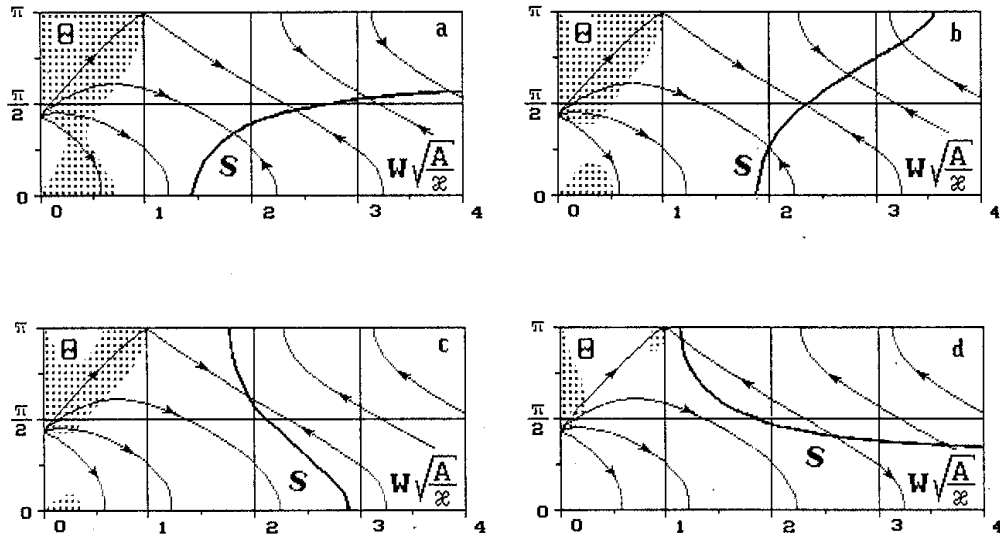


Figure 3: Changes in the parameters of reference regular precession
(a - $A/C = 0.6666$, b - $A/C = 0.9$, c - $A/C = 1.1$, d - $A/C = 1.5$)

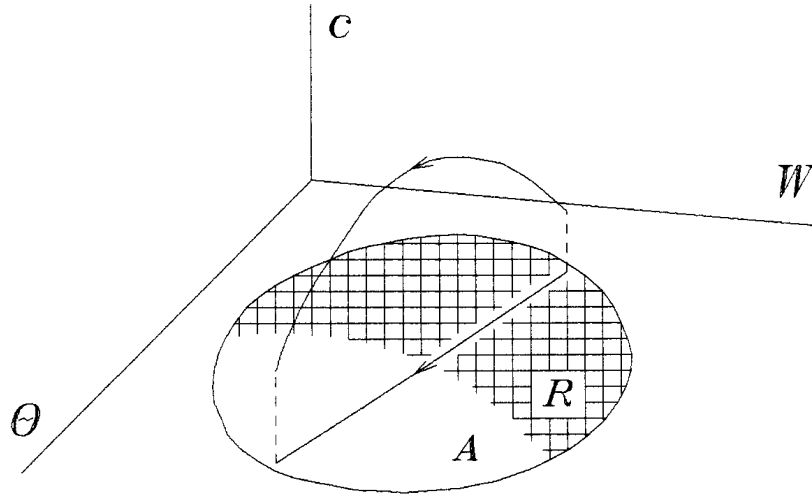


Figure 4: Phase trajectory of averaged system.
A and R are attracting and repelling regions on the manifold $c = 0$

The relations (6.2) define a one-parameter family of stationary motions (curve S on the phase portraits in Fig.3). In these motions the top rolls along the plane without sliding ($V_{PX} = V_{PY} = 0$) . Stability of the rolling without sliding has been investigated in [17].

If geometrical and dynamical parameters of the top satisfy the inequality

$$aC > r|C - A| \quad (6.3)$$

the possible value of the precession velocity in stationary motion is limited:

$$\frac{r\kappa}{aC + r|C - A|} < W^2 < \frac{r\kappa}{aC - r|C - A|}$$

In the case when the inequality (6.3) holds, there exist solutions for which the angle of nutation increases from ≈ 0 to $\approx \pi$ (Fig.3,b,c). These solutions describe the top's overturning.

When the velocity of the precession is large enough, the equation for $\dot{\Theta}$ in (5.1) can be written in the form

$$\dot{\Theta} = \varepsilon mg \sin \Theta (r - a \cos \Theta) \left[\frac{a}{A} - r \cos \Theta \left(\frac{1}{C} - \frac{1}{A} \right) \right] + O\left(\frac{\varepsilon}{W}\right). \quad (6.4)$$

Using (6.4) it is easy to obtain that

$$\arccos [\tanh (C_{\Theta} - d_- t)] \leq \Theta(t) \leq \arccos [\tanh (C_{\Theta} - d_+ t)]$$

where

$$\begin{aligned} C_{\Theta} &= \operatorname{arcth} (\cos \Theta(0)) , \\ d_+ &= \max_{0 \leq \Theta \leq \pi} \Gamma(\Theta) \leq \frac{1}{2} \varepsilon mg (r + a) \left(\frac{a}{A} + r \left| \frac{1}{C} - \frac{1}{A} \right| \right) , \\ d_- &= \min_{0 \leq \Theta \leq \pi} \Gamma(\Theta) \geq \frac{1}{2} \varepsilon mg (r - a) \left(\frac{a}{A} - r \left| \frac{1}{C} - \frac{1}{A} \right| \right) , \\ \Gamma(\Theta) &= \frac{1}{2} \varepsilon mg (r - a \cos \Theta) \left[\frac{a}{A} + r \cos \Theta \left(\frac{1}{C} - \frac{1}{A} \right) \right] \end{aligned}$$

The characteristic time for the top's overturning is estimated by

$$d_+^{-1} < T_{inv} < d_-^{-1} .$$

Finally we examine trivial stationary regimes in which the top rotates uniformly about the axis of dynamical symmetry directed along the vertical. The trivial regimes can be represented by points at the upper and lower boundaries of the phase portrait. The behavior of trajectories near the boundaries are in the agreement with the results of investigations of the stability of trivial regimes [3].

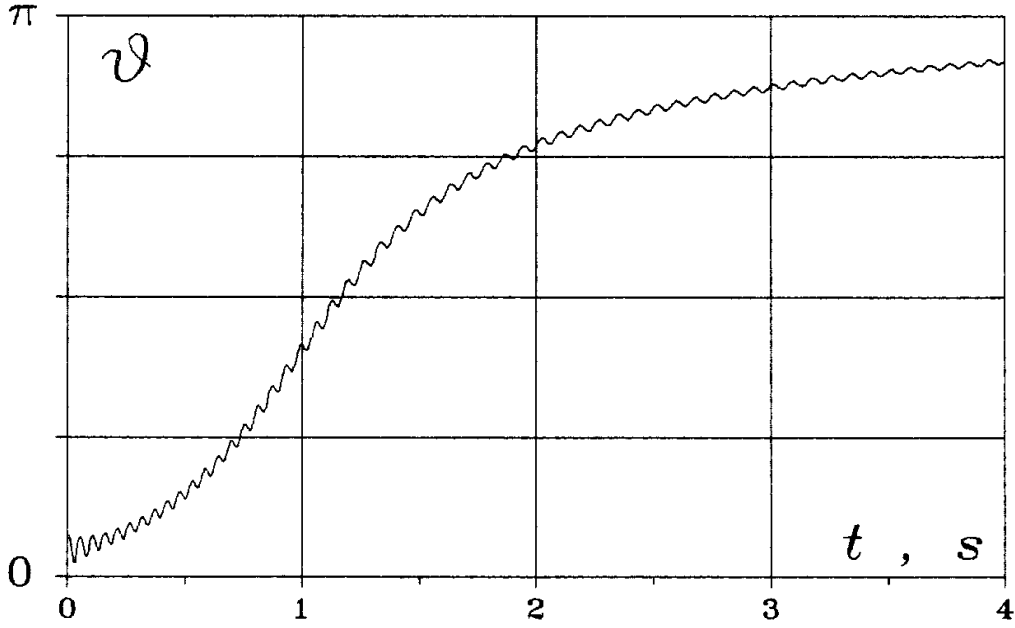


Figure 5: Nutation angle of a rising top

7. Numerical simulation of the motion of the top

Properties of the perturbed motion obtained by the consideration of the averaged system (5.1) are confirmed by results of numerical integration of the original system (2.1). In addition, the numerical simulation reveals some new features of the motion.

Fig.5 shows a typical plot of the nutation angle ϑ with respect to the time t for the overturning top. We have chosen

$$\begin{aligned} A &= 10 \text{ g}\cdot\text{cm}^2, \quad C = 9 \text{ g}\cdot\text{cm}^2, \quad m = 10 \text{ g}, \\ a &= 0.3 \text{ cm}, \quad r = 1.5 \text{ cm}, \quad \varepsilon = 0.005 \text{ s/cm}. \end{aligned}$$

The initial values are

$$V_{GX}(0) = V_{GY} = \Omega_x(0) = \psi(0) = \varphi(0) = 0,$$

$$\Omega_y(0) = 16.695 \text{ rad/s}, \quad \Omega_z(0) = 129.372 \text{ rad/s}, \quad \Theta(0) = 0.241 \text{ rad}.$$

The overturning process lasts approximately four seconds. The trajectory of the point P on the surface of the top is displayed in Fig.6 (we recall that P is the contact point between the top and the plane). In the overturning process this point moves from the lower hemisphere to the upper one. Qualitatively the trajectory can be described as a spiral curve which changes its direction in a vicinity of the equator. The fine spikes and loops are caused by small nutational oscillations of the top. It is not difficult to prove that these spikes and loops are oriented towards the nearest pole of the top in the case $A > C$ and towards the the equator in the case $A < C$. As an example the trajectory of the point P on the surface of a top with oblate ellipsoid of inertia is depicted in fig.7. Here we have chosen

$$A = 10 \text{ g}\cdot\text{cm}^2, \quad C = 11 \text{ g}\cdot\text{cm}^2, \quad m = 10 \text{ g},$$

$$a = 0.3 \text{ cm} , \ r = 1.5 \text{ cm} , \ \varepsilon = 0.009 \text{ s/cm} .$$

The initial values are

$$V_{GX}(0) = V_{GY} = \Omega_x(0) = \psi(0) = \varphi(0) = 0 ,$$

$$\Omega_y(0) = 21.590 \text{ rad/s} , \ \Omega_z(0) = 137.244 \text{ rad/s} , \ \Theta(0) = 0.241 \text{ rad} .$$

It is interesting to compare Fig.6 and Fig.7 with the observed carbon traces after having spun an ordinary plastic top on the smoked glass [4]. Taking into account the orientation of the trajectory loops in the photo of the top in [4], we can conclude that this top has a prolate ellipsoid of inertia.

Conclusions

We present some new qualitative results on motion of the top with spherical shape on a plane with friction. The results are obtained by perturbation analysis of differential equations which describe the dynamics of the top. The special evolutionary variables were shown to be convenient for carrying out the averaging procedure.

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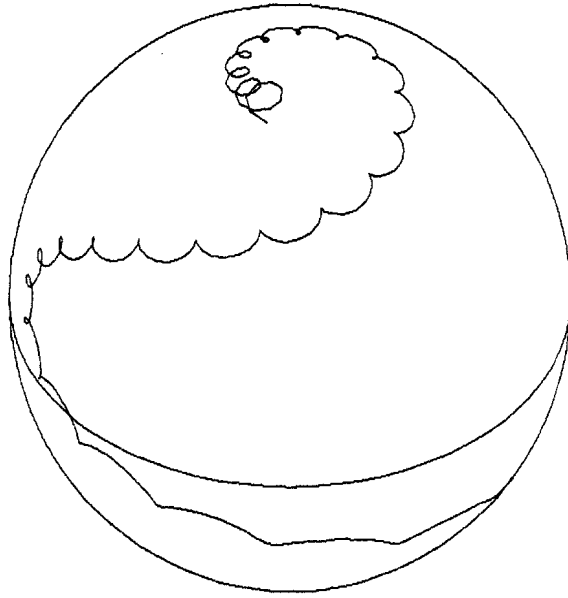


Figure 6: Contact point trajectory in the case $A > C$
The top is overturned

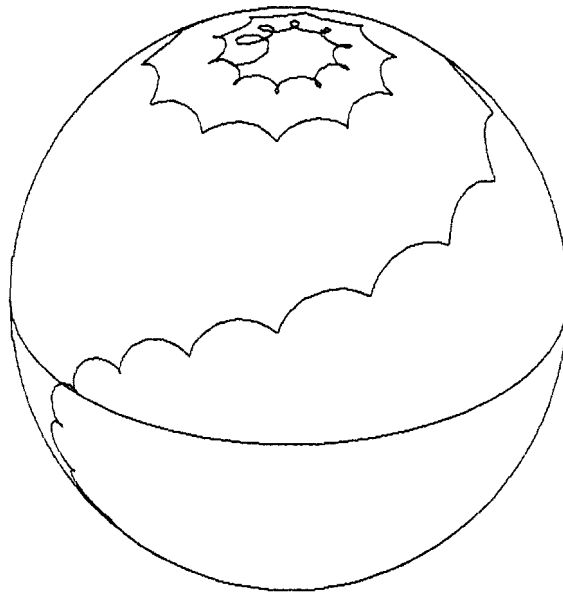


Figure 7: Contact point trajectory in the case $C > A$
The top is overturned

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